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Quantum effects in systems with accelerated mirrors: II. Electromagnetic field

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Abstract. The problem of the electromagnetic vacuum state definition in a part of Minkowski space bounded by a single spherical mirror or two concentric spherical mirrors which expand with a uniform acceleration is considered. The causal Green functions and the vacuum stress–energy tensor for these systems are obtained.

1. Introduction

In the previous paper (Frolov and Serebriany 1979) (paper I) a new method was proposed which allows one to obtain the Green functions of the massless scalar field in the four-dimensional space–time in the presence of the moving conducting boundaries (mirrors) of a special form. This method was applied to the particular cases of a single spherical mirror and a pair of concentric spherical mirrors which expand with uniform acceleration. The main steps of this method are: (i) the introduction of imaginary time $\tau = \mathrm{i}t$; (ii) the demonstration that the corresponding Euclidean Green function vanishing on the analytically continued boundaries coincides with the potential of a unit point electric charge in the four-dimensional space; (iii) the use of the method of images to find this potential; (iv) the definition of the causal Green function in a physical space using the Wick rotation; (v) the reconstruction of the spaces of in- and out-states corresponding to the obtained causal Green function, and (vi) evaluation of the in–in Green function and calculation of the vacuum stress–energy tensor.

In this paper we use this method to define the *electromagnetic* causal Green function (§ 3) and the vacuum stress–energy tensor (§ 4) for the same systems with expanding mirrors which were considered in I, that is in the space outside or inside a single mirror described by the equation Σ_a: $(x)^2 = \boldsymbol{x}^2 - t^2 = a^2$ (problem A) and in the cavity between two concentric spherically symmetric mirrors Σ_a and Σ_b described by the equations Σ_a: $(x)^2 = a^2$, Σ_b: $(x)^2 = b^2$, $a < b$, (problem B). A detailed discussion of these systems together with the notation used in this paper can be found in I. The corresponding vacuum states are shown (in the Appendix) to be stable. To get over the difficulties arising because of the gauge group presence some general formulae for the Green functions in the gauge invariant theories are given and the vacuum stability condition in the gauge invariant form is obtained in § 2.

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2. The Green functions for the gauge-invariant theory in an external field

The theory we are interested in can be described by an action

$$S[\phi]=\tfrac{1}{2}\phi^i S_{ij}\phi^j \tag{2.1}$$

where ϕ^i are dynamical fields and S_{ij} is a self-adjoint second-order linear differential operator, the coefficients of which are dependent on some external field. From now on the DeWitt compact notations are used (DeWitt 1965). The general formulae of this section are specified and used for the problem under consideration in the following sections. If

$$S_{ij}=P^{\mu\nu}_{ij}\delta_{,\mu\nu}(x,x')-(N^{\mu}_{ij}-P^{\mu\nu}_{ij,\nu})\delta_{,\mu}(x,x')-(T_{ij}+\tfrac{1}{2}N^{\mu}_{ij,\mu})\delta(x,x')$$

$$P^{\mu\nu}_{ij}=P^{\nu\mu}_{ij}=P^{\mu\nu}_{ji},\qquad N^{\mu}_{ij}=-N^{\mu}_{ji},\qquad T_{ij}=T_{ji},$$

then the canonical bilinear form

$$B(\phi,\psi)\equiv\phi^i b_{ij}\psi^j=\int_C \mathrm{d}\Sigma_\mu\,[\phi^i P^{\mu\nu}_{ij}\psi^j_{,\nu}-\phi^i_{,\nu}P^{\mu\nu}_{ij}\psi^j-\phi_i N^{\mu}_{ij}\psi^j] \tag{2.2}$$

defined for any two solutions ϕ^i and ψ^j of the field equations does not depend on the particular choice of the total Cauchy surface C.

We suppose that the action (2.1) is invariant under the Abelian gauge group of transformations $\delta\phi^i=R^i_\alpha\delta\xi^\alpha$, where R^i_α are the generators and $\xi^\alpha(x)$ are the group parameters. In this case the field equations and the form B are degenerate, that is $S_{ij}R^j_\alpha=0$ and $B(\phi,R^i_\alpha\delta\xi^\alpha)=0$ for an arbitrary solution ϕ^i. To single out the physical modes following DeWitt (1965) we impose the supplementary conditions

$$R_{i\alpha}\phi^i=0,\qquad \bar{R}_{i\alpha}\phi^i=0 \tag{2.3}$$

where $R_{i\alpha}=\gamma_{ij}R^j_\alpha$ and γ_{ij} is a completely local continuous symmetric matrix of such a form that the symmetric operator $F_{\alpha\beta}=R_{i\alpha}R^i_\beta$ is non-singular and $\bar{R}_{i\alpha}$ satisfies the equation $\bar{R}^i_\alpha\gamma_{ij}\bar{R}^j_\beta=F_{\alpha\beta}$. In the space $\mathscr{R}$ of solutions satisfying (2.3) the form B is non-degenerate. Denote by u the basis (u_A,u^*_A) in $\mathscr{R}$ normalised as follows: $B(u_A,u_B)=0$, $B(u^*_A,u_B)=\mathrm{i}\delta_{AB}$. The quantum field $\boldsymbol{\phi}$ may be written in the form

$$\boldsymbol{\phi}^i=\boldsymbol{a}_A u^i_A+\overset{+}{\boldsymbol{a}}_A u^{*i}_A+R^i_\alpha\boldsymbol{\xi}^\alpha \tag{2.4}$$

where $\overset{+}{\boldsymbol{a}}_A=\mathrm{i}B(u_A,\boldsymbol{\phi})$ and $\boldsymbol{a}_A=-\mathrm{i}B(u^*_A,\boldsymbol{\phi})$ are the operators of the creation and annihilation of particles in a physical mode u_A and $\boldsymbol{\xi}^\alpha$ are arbitrary Hermitian functions of the $\boldsymbol{a}_A$, $\overset{+}{\boldsymbol{a}}_A$. The usual form of the commutation relations for $\boldsymbol{a}_A$ and $\overset{+}{\boldsymbol{a}}_A$ is a simple consequence of the following gauge invariant form of the commutation relations for the field $\boldsymbol{\phi}^i$:

$$[B(\boldsymbol{\phi},f_1),B(\boldsymbol{\phi},f_2)]=-\mathrm{i}B(f_1,f_2)$$

which is to be held for arbitrary C-number solutions f^i_1 and f^i_2. The field $\boldsymbol{\phi}^i$ is said to be taken in the $\mathscr{R}$ gauge if $\boldsymbol{\xi}^\alpha$ in equation (2.4) is put equal to zero.

Now consider the situation when two bases $_+U$ and $_-U$ are given which are determined by an appropriate positive frequency choice in the future ($_+U$) and in the past ($_-U$). The corresponding in- and out-vacuum states are defined as follows:

$$_+\boldsymbol{a}_A|\text{out, vac}\rangle\equiv-\mathrm{i}B(_+u^*_A,\boldsymbol{\phi})|\text{out, vac}\rangle=0,$$

$$_-\boldsymbol{a}_A|\text{in, vac}\rangle\equiv-\mathrm{i}B(_-u^*_A,\boldsymbol{\phi})|\text{in, vac}\rangle=0.$$

Two Green functions are of particular interest $G^{ij'}_{(\mathrm{in})} = \mathrm{i}\langle \mathrm{in, vac}|T(\phi^i\phi^{j'})|\mathrm{in, vac}\rangle$ and $G^{ij'} = S_0^{-1}\langle \mathrm{out, vac}|T(\phi^i\phi^{j'})|\mathrm{in, vac}\rangle$, where $S_0 = \langle \mathrm{out, vac}|\mathrm{in, vac}\rangle$. The first one can be used to find the in-vacuum expectation value. The second function is a causal (Feynman) Green function which contains all the necessary information about the linear quantum system. These Green functions are evidently not gauge invariant. If the field ϕ is taken in the $\mathscr{R}$ gauge then we refer to $G^{ij'}_{(\mathrm{in})}$ and $G^{ij'}$ as the Green functions in $\mathscr{R}$ gauge. The difference between the Green functions taken in two different gauges is a pure gauge term of the form $D^{ij'} = R^{j'}_{\alpha}p^{\alpha i} + q^{\alpha i}R^{j'}_{\alpha} + R^{i}_{\alpha}t^{\alpha\beta'}R^{j'}_{\beta'}$. For any two gauge invariant functionals A and B ($A_{,i}R^i_\alpha = B_{,i}R^i_\alpha = 0$) the expressions $A_{,i}G^{ij'}_{(\mathrm{in})}B_{,j'}$ and $A_{,i}G^{ij'}B_{,j'}$ are gauge independent.

To find an explicit expression for $G^{ij'}$ it is convenient to introduce a new mixed basis ${}_+U_-$: $({}_+u^i_A, {}_-u^{*i}_A)$ obeying the normalisation conditions $B({}_+u_A, {}_+u_B) = B({}_-u^*_A, {}_-u^*_B) = 0$, $B({}_-u^*_A, {}_+u_B) = \mathrm{i}u_{AB}$. Using this non-orthogonal basis one can write in the $\mathscr{R}$ gauge

$$\phi^i = {}_+a^\dagger \mathscr{A}^{-1}{}_-u^{*i} + {}_+u^i\mathscr{A}^{-1}{}_-a,$$

$$G^{ij'} = \mathrm{i}[\theta(i, j')_+u^i\mathscr{A}^{-1}{}_-u^{*j'} + \theta(j', i)_+u^{j'}\mathscr{A}^{-1}{}_-u^{*i}].$$

To find the vacuum stability condition we notice that in the $\mathscr{R}$ gauge for $i < j' < m''$ the equality $P^{im''} \equiv G^{ij'}b_{j'k'}G^{*k'm''} \equiv {}_+u^i\mathscr{A}^{-1}B({}_-u^*, {}_+u^*)\mathscr{A}^{-1*}{}_-u^{m''} = 0$ is equivalent to $B({}_-u^*_A, {}_+u^*_B) = 0$, that is it is a necessary and sufficient condition for vacuum stability. Using the gauge invariance property of the form B one can verify that this vacuum stability condition can be written in the gauge-invariant form

$$A_{,i}G^{ij'}b_{j'k'}G^{*k'm''}B_{,m''} = 0, \qquad i < j' < m'' \tag{2.5}$$

which is to be satisfied for an arbitrary pair of invariants A and B. If the vacuum is stable then $G^{ij'}_{(\mathrm{in})} = G^{ij'}$. In the opposite case, to express $G^{ij'}_{(\mathrm{in})}$ in terms of $G^{ij'}$ one must resolve a system of equations analogous to equations (A1.8)–(A1.10) of paper I.

3. The electromagnetic Green functions

In this paper we restrict ourselves by considering the vacuum stress–energy tensor for the electromagnetic field a_μ. The action $S[a] = -\frac{1}{4}\int f_{\mu\nu}f^{\mu\nu}\,\mathrm{d}^4x$, where $f_{\mu\nu} = a_{\nu,\mu} - a_{\mu,\nu}$ provides the Maxwell equations $f^{\mu\nu}_{,\nu} = 0$. The boundary conditions on the moving mirror surface Σ are $\epsilon^{\mu\nu\rho\sigma}f_{\mu\nu}(x)\xi_\rho(x)|_{x\in\Sigma} = 0$, where ξ^ρ is a vector normal to Σ. These boundary conditions may be considered as a special type of the external field. The action and the boundary conditions are invariant under gauge transformations $a_\mu \to a_\mu + \partial_\mu\chi$. This theory is obviously a particular case of the general theory considered in the previous section with the identifications

$$i, j \to \lambda, \rho, \qquad \phi^i \to a_\lambda, \qquad R^i_\alpha \to R_\lambda = -\delta_{,\lambda}(x, x')$$

$$P^{\mu\nu}_{ij} \to P^{\lambda\rho;\mu\nu} = (-g)^{1/2}(g^{\lambda\rho}g^{\mu\nu} - \tfrac{1}{2}g^{\lambda\mu}g^{\rho\nu} - \tfrac{1}{2}g^{\lambda\nu}g^{\rho\mu})$$

$$N^\mu_{ij} = 0, \qquad T_{ij} = 0.$$

The canonical bilinear from B can be written as

$$B(\underset{1}{a}, \underset{2}{a}) = \int_C (\underset{1}{a}_\alpha \underset{2}{f}_{\nu\beta} - \underset{2}{a}_\alpha \underset{1}{f}_{\nu\beta})g^{\alpha\beta}g^{\mu\nu}(-g)^{1/2}\,\mathrm{d}\Sigma_\mu$$

where C is a total Cauchy surface in the problem under consideration.

Consider the inversion transormation $x \to \tilde{x} = \boldsymbol{J}_a x = (a^2/(x)^2)x$ and define $\tilde{a}_\mu(\tilde{x}) \equiv (\boldsymbol{J}a)_\mu(\tilde{x}) = (\partial x^\nu/\partial \tilde{x}^\mu)a_\nu(x)$. Using the conformal invariance of the Maxwell equations one can show that if $a_\mu(x)$ is a solution of the free equations then $\tilde{a}_\mu(\boldsymbol{J}a)_\mu$ is also a solution. Moreover the solution $\hat{a}_\mu = a_\mu - \tilde{a}_\mu$ satisfy the boundary conditions $\epsilon^{\mu\nu\rho\sigma}\hat{f}_{\mu\nu}\xi_\rho|_{\Sigma a} = 0$. It should be emphasised that if the solution a_μ is taken in a particular $\mathscr{R}$ gauge then in general $\tilde{a}_\mu$ does not satisfy $\mathscr{R}$ gauge conditions and to have the solution $\hat{a}_\mu$ in the $\mathscr{R}$ gauge one needs to add a pure gauge term.

In the same manner one can show that if $G^{(0)}_{\mu\nu}$ is a free causal Green function in a given $\mathscr{R}$ gauge then the Green function for the problem A in the $\mathscr{R}$ gauge can be written

$$\begin{aligned} G_{\mu\nu'}(x|x') &= (\boldsymbol{I} - \boldsymbol{J}_a)G^{(0)}_{\mu\nu'}(x|x') + (\text{a pure gauge term}), \\ (\boldsymbol{J}_a G^{(0)})_{\mu\nu'}(x|x') &= (\partial\tilde{x}^\lambda/\partial x^\mu)G^{(0)}_{\lambda\nu'}(\tilde{x}|x'). \end{aligned} \tag{3.1}$$

In the case of the problem B the corresponding Green function is

$$G_{\mu\nu'}(x|x') = (\boldsymbol{I} - \boldsymbol{K})G^{(0)}_{\mu\nu'}(x|x') + (\text{a pure gauge term}) \tag{3.2}$$

where

$$\boldsymbol{K} = (\boldsymbol{I} - \boldsymbol{J}_a\boldsymbol{J}_b)^{-1}(\boldsymbol{J}_a - \boldsymbol{J}_a\boldsymbol{J}_b) + (\boldsymbol{I} - \boldsymbol{J}_b\boldsymbol{J}_a)^{-1}(\boldsymbol{J}_b - \boldsymbol{J}_b\boldsymbol{J}_a)$$

In the Appendix it is shown that these Green functions satisfy the vacuum stability condition (2.5).

4. Electromagnetic vacuum stress–energy tensor.

The electromagnetic strength tensor $f_{\mu\nu} = a_{\nu,\mu} - a_{\mu,\nu} \equiv F^{\alpha'}_{\mu\nu}a_{\alpha'}$ is gauge invariant. We may use this invariant to determine the gauge-invariant Green function $G_{\mu\nu;\lambda'\gamma'}(x|x') = F^{\alpha''}_{\mu\nu}G_{\alpha''\beta'''}F^{\beta'''}_{\lambda'\gamma'}$. If the mirrors are absent $G^{(0)}_{\mu\nu;\lambda'\gamma'}$ satisfies the equation

$$(\partial/\partial x^\mu)G^{(0)\mu\nu;\lambda'\gamma'}(x|x') = (\eta^{\nu\lambda}\partial^{\gamma'} - \eta^{\nu\gamma}\partial^{\lambda'})\delta(x - x') \tag{4.1}$$

and can be written in the form ($\partial_\mu \equiv \partial/\partial x^\mu$, $\partial_{\lambda'} \equiv \partial/\partial x'^\lambda$)

$$\begin{aligned} G^{(0)}_{\mu\nu;\lambda'\gamma'}(x|x') &= \mathrm{d}_{\mu\nu;\lambda'\gamma'}G^{(0)}(x|x') \\ \mathrm{d}_{\mu\nu;\lambda'\gamma'}(x|x') &= \eta_{\nu\gamma'}\,\partial_\mu\partial_{\lambda'} + \eta_{\mu\lambda}\,\partial_\nu\partial_{\gamma'} - \eta_{\nu\lambda'}\,\partial_\mu\partial_{\gamma'} - \eta_{\mu\gamma'}\,\partial_\nu\partial_{\lambda'} \\ G^{(0)}(x|x') &= \frac{\mathrm{i}}{4\pi^2}\,\frac{1}{(x - x')^2 + \mathrm{i}\epsilon} \end{aligned} \tag{4.2}$$

The invariant Green functions calculated for the problems A and B can be presented in the form

$$G_{\mu\nu;\lambda'\gamma'} = (\boldsymbol{I} - \boldsymbol{K})G^{(0)}_{\mu\nu;\lambda'\gamma'} \tag{4.3}$$

where

$$\begin{aligned} &\boldsymbol{K} = \boldsymbol{J}_a && \text{(problem A)} \\ &\boldsymbol{K} = \boldsymbol{J}_a + \boldsymbol{J}_b - \boldsymbol{J}_a\boldsymbol{J}_b - \boldsymbol{J}_b\boldsymbol{J}_a + \boldsymbol{J}_a\boldsymbol{J}_b\boldsymbol{J}_a + \boldsymbol{J}_b\boldsymbol{J}_a\boldsymbol{J}_b - \ldots && \text{(problem B)} \end{aligned} \tag{4.4}$$

and the action of the inversion operator $\boldsymbol{J}_a$ is defined as

$$\begin{aligned} (\boldsymbol{J}_a G)^{\mu\nu;\lambda'\gamma'}(\tilde{x}|x') &= \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\,\frac{\partial x^\nu}{\partial \tilde{x}^\beta}\,G^{\alpha\beta;\lambda'\gamma'}(x|x') \\ &= \tilde{d}^{\mu\nu;\lambda'\gamma'}(\tilde{x}|x')\boldsymbol{J}_a(x')G^{(0)}(\tilde{x}|x') \end{aligned}$$

where

$$\tilde{d}_{\mu\nu;\lambda'\gamma'}(\tilde{x}|x') = \frac{\partial^2}{\partial\tilde{x}^\mu\,\partial x'^\lambda}\tilde{\eta}_{\nu\gamma'}(x') + \frac{\partial^2}{\partial\tilde{x}^\nu\,\partial x'^\gamma}\tilde{\eta}_{\mu\lambda'}(x') - \frac{\partial^2}{\partial\tilde{x}^\nu\,\partial x'^\lambda}\tilde{\eta}_{\mu\gamma'}(x')$$

$$-\frac{\partial^2}{\partial\tilde{x}^\mu\,\partial x'^\gamma}\tilde{\eta}_{\nu\lambda'}(x') \qquad \tilde{\eta}_{\mu\nu'}(x') = \eta_{\mu\nu'} - 2\frac{x'_\mu x'_\nu}{(x')^2}$$

$$\boldsymbol{J}_a(x')\boldsymbol{G}^{(0)}(\tilde{x}|x') = \frac{\mathrm{i}}{4\pi^2}\frac{a^2}{(x')^2}\frac{1}{(\tilde{x}-a^2/(x')^2x')^2 + \mathrm{i}\epsilon\,\mathrm{sign}(x')^2}$$

We do not indicate in (4.3) on which point the operator of inversion acts. This is possible because of the following property of the Green function

$$\boldsymbol{J}_a(x)\boldsymbol{G}_{\mu\nu;\lambda'\gamma'}(x|x') = \boldsymbol{J}_a(x')\boldsymbol{G}_{\mu\nu;\lambda'\gamma'}(x|x')$$

The Green functions (4.3) and (4.4) satisfy the boundary conditions

$$\epsilon^{\mu\nu\rho\sigma}G_{\mu\nu;\alpha'\beta'}(x|x')\xi_\rho(x)|_{x\in\Sigma} = \epsilon^{\alpha'\beta'\gamma'\delta'}G_{\mu\nu;\alpha'\beta'}(x|x')\xi_{\gamma'}(x')|_{x'\in\Sigma} = 0.$$

They can be written in the explicit form

Problem A:

$$G_{\mu\nu;\alpha'\beta'}(x|x') = \frac{\mathrm{i}}{4\pi^2}\left[\mathrm{d}_{\mu\nu;\alpha'\beta'}(x|x')\frac{1}{(x-x')^2+\mathrm{i}\epsilon} - \tilde{\mathrm{d}}_{\mu\nu;\alpha'\beta'}(x|x')\frac{a^2}{x^2x'^2 - 2a^2xx' + a^4 + \mathrm{i}\epsilon}\right]$$

Problem B:

$$G_{\mu\nu;\alpha'\beta'}(x|x') = \frac{\mathrm{i}}{4\pi^2}\sum_{n=-\infty}^{\infty}\left(\frac{a}{b}\right)^{2n}\left[\mathrm{d}_{\mu\nu;\alpha'\beta'}(x|x')\frac{1}{((a/b)^{2n}x - x')^2 + \mathrm{i}\epsilon}\right.$$

$$\left. - \tilde{\mathrm{d}}_{\mu\nu;\alpha'\beta'}(x|x')\frac{a^2}{(x)^2}\frac{1}{\{(a/b)^{2n}[a^2/(x)^2]x - x'\}^2 + \mathrm{i}\epsilon\,\mathrm{sign}(x)^2}\right]$$

Because of the vacuum stability property the vacuum stress–energy tensor can be defined by a relation

$$T^{\mu\nu}(x) = \langle \mathrm{in, vac}|T^{\mu\nu}_{\mathrm{reg}}(x)|\mathrm{in, vac}\rangle$$

$$= -\mathrm{i}\lim_{x'\to x}(\delta^\mu_\alpha\delta^{\nu'}_{\beta'} - \tfrac{1}{4}\eta^{\mu\nu'}\eta_{\alpha\beta'})G^{\alpha\lambda;\beta'}_{(\mathrm{reg})\lambda'}(x|x')$$

The subscript 'reg' means that the vacuum expectation value in the space without mirrors is subtracted. The calculations give

Problem A:

$$T^{\mu\nu}(x) = 0 \tag{4.5}$$

Problem B:

$$\begin{aligned} T^{\mu\nu}(x) &= -\frac{4}{\pi^2(x)^6}(x^\mu x^\nu - \tfrac{1}{4}(x)^2\eta^{\mu\nu})t(a,b) \\ t(a,b) &= \sum_{n=1}^{\infty}\frac{(a/b)^{2n} + (b/a)^{2n}}{[(a/b)^n - (b/a)^n]^4} \end{aligned} \tag{4.6}$$

These results are very similar to the corresponding results of the paper I for the improved stress–energy tensor of the scalar massless field. The only difference is the value of the constant $t(a, b)$.

When the acceleration of both mirrors in the problem B tends to zero and the distance between them is fixed, we obtain two parallel plane mirrors. Using the coordinate shift $x^\mu \to x^\mu - \delta^\mu_1 a$ and finding the limit of equation (4.6) when $a, b \to \infty$, $\Delta = b - a =$ constant, one has

$$T^{\mu\nu}(x) = -\frac{\pi^2}{180\Delta^4}(\delta^\mu_1\delta^\nu_1 - \tfrac{1}{4}\eta^{\mu\nu}),$$

which reproduces the results obtained by Brown and Maclay (1969).

Appendix. Proof of the vacuum stability condition

Let the equation $D_g[\phi] = 0$ for a field ϕ^i be invariant under conformal ($g \to \hat{g} = \omega^2 g$, $\phi \to \hat{\phi} = \hat{\phi}(\omega, \phi)$) and general coordinate transformations. Then the corresponding canonical bilinear form B also possesses this property (Frolov 1979). Now consider this equation in a flat space–time. If $x \to \tilde{x} = \boldsymbol{J}x = (a^2/(x)^2)x$ is the inversion transformation ($\mathrm{d}x_\mu\, \mathrm{d}x^\mu = \omega^2\, \mathrm{d}\tilde{x}^\mu$, $\omega = a^2/(x)^2$) and $\phi \to \tilde{\phi} = \boldsymbol{J}\phi$ is the corresponding transformation of the field (i.e. the transformation induced by the inversion $x \to \boldsymbol{J}x$ mapping combined with an appropriate conformal transformation with ω as a conformal factor), then one has $B_{\tilde{U}}(\tilde{\phi}_1, \tilde{\phi}_2) = B_U(\phi_1, \phi_2)$, where the subscript indicates the three-dimensional region of the integration ($\tilde{U} = \boldsymbol{J}U$). Here we restrict ourselves by considering the exterior problem A. The vacuum stability proof in the other cases can be easily given using the same method.

In the exterior problem A one can take as U the part of the $t = 0$ plane R^3: $U = \{\boldsymbol{x}: |\boldsymbol{x}|^2 \geq a^2\}$. In this case $(\boldsymbol{I} + \boldsymbol{J})U = R^3$. Using the property of the inversion operator $\boldsymbol{J}^2 = \boldsymbol{I}$ one has

$$\begin{aligned} B_U(\phi_1 - \tilde{\phi}_1, \phi_2 - \tilde{\phi}_2) &= B_{\tilde{U}}(\tilde{\phi}_1 - \phi_1, \tilde{\phi}_2 - \phi_2) \\ &= \tfrac{1}{2}B_{R^3}(\phi_1 - \tilde{\phi}_1, \phi_2 - \tilde{\phi}_2) = B_{R^3}(\phi_1 - \tilde{\phi}_1, \phi_2) \end{aligned} \tag{A1}$$

These relations and the vacuum stability in a Minkowski space imply that the vacuum stability condition in the problem under consideration is equivalent to the following condition

$$\begin{aligned} \Pi_{\alpha_1\alpha_2;\mu''_1\mu''_2} \equiv D^{\mu''}_{\mu''_1\mu''_2} D^{\alpha}_{\alpha_1\alpha_2} \int_{\substack{x^{0\prime}=0 \\ \boldsymbol{x}' \in R^3}} & [H^{\beta'}_{\alpha}(x|x')\partial_{[\beta'}\bar{G}^{(0)}_{\gamma']\mu''}(x'|x'') \\ & - \bar{G}^{(0)\beta'}_{\mu''}(x'|x'')\,\partial_{[\beta'}H_{|\alpha|\gamma']}(x|x')]\,\mathrm{d}\Sigma^{\gamma'} = 0 \end{aligned} \tag{A2}$$

which is to be satisfied for arbitrary points x, x'' with $x^0 < 0 < x''^0$. Here $H_{\alpha\beta'} = (\boldsymbol{J}G^{(0)})_{\alpha\beta'}(x|x') \equiv (\delta^{\sigma'}_{\beta'} - 2(x'^{\sigma'}x'_{\beta'}/(x')^2))\,(a^2/(x')^2)G^{(0)}_{\alpha\sigma'}(x|a^2/(x')^2x')$ and $D^{\alpha}_{\alpha_1\alpha_2} \equiv \delta^{\alpha}_{\alpha_1}\,\partial_{\alpha_2} - \delta^{\alpha}_{\alpha_2}\,\partial_{\alpha_1}$. Because of the gauge invariance of the vacuum stability condition (2.5) it is enough to verify equation (A2) taking the electromagnetic Green function $G^{(0)}_{\alpha\beta'}$ in a radiative gauge ($a_0 = 0$, div $\boldsymbol{a} = 0$):

$$G^{(0)}_{\alpha\beta'} = \delta^a_\alpha \delta^{b'}_{\beta'}\delta_{ab'}G^{(0)}(x|x') + \ldots, \qquad a, b = 1, 2, 3. \tag{A3}$$

The $G^{(0)}(x|x')$ is a causal Green function of the scalar field. The ellipsis denotes pure gauge terms which do not contribute to $\Pi_{\alpha_1\alpha_2;\mu_1''\mu_2''}$. Substituting equation (A3) into equation (A2), after calculation one has

$$\begin{aligned}\Pi_{\alpha_1\alpha_2;\mu_1''\mu_2''} = {} & 2D^{\alpha}_{\alpha_1\alpha_2}D^{\mu''}_{\mu_1''\mu_2''}\boldsymbol{J}_a(x)(\partial_0-\partial_0'')\delta^{i'}_{\mu''}\delta^{\rho'}_{\alpha} \\ & \times\int \mathrm{d}^3x'\frac{x'_{i'}x'_{l'}}{|\boldsymbol{x}'|^2}G^{(0)}(x|x')\bar{G}^{(0)}(x'|x'')\end{aligned} \qquad \text{(A4)}$$

where $\boldsymbol{J}_a(x)f(x)\equiv(a^2/(x)^2)f((a^2/(x)^2)x)$. If $(x-x'')^2<0$ $((x-x'')^2>0)$ then using the Lorentz tranformation one can put $\boldsymbol{x}=\boldsymbol{x}''=0$ $(x^0\to-0,\ x^{0''}\to+0)$. In both cases the integration in (A4) can be fulfilled in an explicit way and one can verify that the vacuum stability condition $\Pi_{\alpha_1\alpha_2;\mu_1''\mu_2''}=0$ is satisfied.

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